# A Product-form C-logit Stochastic Traffic Assignment Model

Ning Lv
School of Traffic and Transportation, Beijing Jiaotong University, Beijing, China
19120872@bjtu.edu.cn

Wenyi Zhang*
School of Traffic and Transportation, Beijing Jiaotong University, Beijing, China, wyzhang@bjtu.edu.cn

Jihui Ma
School of Traffic and Transportation, Beijing Jiaotong University, Beijing, China
jhma@bjtu.edu.cn

## ABSTRACT

As an important member of the stochastic traffic assignment models, the C-logit model has been widely applied in traffic network analysis as well as the development of various modern intelligent traffic managements. It provides a simple solution to address the path overlapping problem existed in classical logit model, while inheriting the simplicity of the latter. However, it is shown in this paper that the C-logit model can easily degrade to the logit model when heavy (or even mild) traffic congestion emerges on a road network. Such degeneracy might stem from the summation formulation of the measurable path cost in the C-logit model. For this reason, this paper suggests a revised (i.e., product-form) formulation for measurable path cost, which is able to resist such degeneracy. The resultant new C-logit model is expected to provide a more precise option for stochastic road network modeling and analysis.

## CCS CONCEPTS

• **Applied computing**; • **Operations research**; • **Transportation**;

## KEYWORDS

Urban transport, Traffic assignment, Stochastic user equilibrium, C-logit model

**ACM Reference Format:**
Ning Lv, Wenyi Zhang*, and Jihui Ma. 2021. A Product-form C-logit Stochastic Traffic Assignment Model. In *The 5th International Conference on Computer Science and Application Engineering (CSAE 2021), October 19–21, 2021, Sanya, China.* ACM, New York, NY, USA, 4 pages. https://doi.org/10.1145/3487075.3487176

## 1 INTRODUCTION

Traffic assignment is aimed to assign travel demands onto a road network, and obtain the resultant network traffic pattern. This work is the basis for scientific traffic network analysis, and is very important for modern intelligent traffic management. In general, the traffic assignment theory has two subdivisions, i.e., the deterministic theory and the stochastic theory. Wardrop [1] first formally presented two basic principles for deterministic traffic assignment, which are now called as user equilibrium (UE) and system optimization (SO). Beckmann et al. [2] proposed a convex mathematical programming model which is equivalent to UE, and formally laid the theoretical foundation for urban transport network analysis [3]. Later, Daganzo and Sheffi [4] introduced the stochastic traffic assignment theory based on the discrete choice theory (see Ben-akiva and Lerman [5] for detailed introduction on the theory and application of the discrete choice analysis in travel demand modeling), where the classical logit model and the probit model were presented. Because of simple structure and closed expression, the former has been widely used in subsequent research. However, the Independence of Irrelevant Alternatives (IIA) hypothesis made in the classic logit model has always been criticized [6]. In order to overcome this problem, Cascetta et al. [7] proposed a C-logit model. In such a C-logit model, path overlapping is formulated as an additional cost to the intrinsic path cost. Then, heavier overlap would decline the quality of a path, and yield a smaller choice probability. The C-logit model provides an applicable method to deal with the IIA criticism, while maintaining the simplicity of the classical logit model. For this reason, the C-logit model has become a competitive member of the logit family. Zhou et al. [8] developed an equivalent mathematical programming formulation for the C-logit model. Except from the C-logit model, other improvements over the logit model still include, but not limit to, the path size logit model [9], the cross-nested logit model [10], and the paired combinatorial logit model [11].

Indeed, the C-logit model possesses a simple and intuitive structure, and it does overcome some deficiencies found in the logit model. However, the model applies an additive method to formulate the resultant loss caused by path overlap, where the path correlation factor is a fixed parameter and cannot vary against either network or traffic condition. This makes it easy to degenerate into the classic logit model when heavy (or even mild) traffic congestion emerges, and reduces the accuracy of the traffic assignment outcome hence. In addition, it is found that the path natural cost and the path correlation factor have different dimensions, and the conversion coefficient between them is hard to determine. In view of these, this paper suggests a slightly revised C-logit stochastic traffic assignment model, which formulates the measurable path cost in a product (rather than additive) form. The numerical example performed in this paper indicates that this revision is effective.

The remainder of this paper is organized as follows. Section 2 presents the product-form C-logit model. Numerical example is presented in Section 3 to compare the traditional model and the current model. Section 4 concludes this paper, and suggests some valuable research directions in the future.


*CSAE 2021, October 19–21, 2021, Sanya, China*

ACM ISBN 978-1-4503-8985-3/21/10…$15.00
https://doi.org/10.1145/3487075.3487176

## 2 PRODUCT-FORM C-LOGIT MODEL

Consider a strongly connected traffic network. Let $A$ and $W$ respectively denote the sets of all edges and origin-destination (OD) pairs in the network. Let $a$ and $w$ be the indices of an edge and an OD pair, respectively. Let $R^w$ denote the set of acyclic paths connecting OD pair $w$, where $r$ is the index of a path.

In a stochastic traffic network, the perceived general cost (random variable) of a path can be expressed as follows:

$$U_r^w = G_r^w + \varepsilon_r^w \ \forall r \in R^w, w \in W \tag{1}$$

In formula (1), $U_r^w$ is a traveler's perceived general cost of path $r$ connecting OD pair $w$, $G_r^w$ is the measurable part of $U_r^w$, and $\varepsilon_r^w$ is the random part. Let $E(\varepsilon_r^w) = 0$ where $E(\cdot)$ indicates the expectation of a random variable.

Based on formula (1), below we give the definition of the path choice probability.

$$P_r^w = \Pr(U_r^w \leq U_i^w, i \in R^w, i \neq r) \ \forall r \in R^w, w \in W \tag{2}$$

In formula (2), $P_r^w$ is the choice probability of path $r$ within OD pair $w$, and $\Pr(\cdot)$ indicates the probability of a random event.

Given that the travel demands are constant and according to the stochastic traffic assignment theory, the proportion of traffic demand assigned to a specific path equals to the choice probability of the path when the road network reaches the stochastic user equilibrium (SUE) state [6]. The corresponding mathematical description is as follows:

$$f_r^w = P_r^w \cdot q^w \ \forall r \in R^w, w \in W \tag{3}$$

In formula (3), $f_r^w$ denotes the flow on path $r$ linking OD pair $w$; and $q^w$ is the travel demand of OD pair $w$.

Let the random part $\varepsilon_r^w$ in formula (1) follow independent and identical Gumbel distribution. The path choice probability shown in equation (2) can then be expressed by

$$P_r^w = \frac{\exp(-\theta G_r^w)}{\sum_{i \in R^w} \exp(-\theta G_i^w)} \ \forall r \in R^w, w \in W \tag{4}$$

In above formula, $\theta$ is a non-negative constant (reflecting the perfection extent of information), and $\exp(\cdot)$ is an exponential function operator with Euler's constant as base.

Let $G_r^w = C_r^w \ \forall r \in R^w, w \in W$ (where $C_r^w$ is the measurable path travel time in formula (1)), and we obtain the classic Logit model. Let $G_r^w = C_r^w + \kappa\varphi_r^w \ \forall r \in R^w, w \in W$ (where, $\kappa$ is a non-negative constant, and $\varphi_r^w$ represents the overall overlap degree between path $r$ and others connecting OD pair $w$), that is, the measurable general cost of a path is the summation of the measurable travel time and the resultant overlap loss, and the traditional C-logit model can be obtained. Obviously, the measurable path travel time in the C-logit model would change with network topology and traffic demand, but the overlap loss remains unchanged. Then, when traffic congestion intensifies, the measurable general path cost would be dominated by the increased path travel time, and the effect of overlap loss would diminish. As a result, the C-logit model would degrade to the logit model. To overcome this problem, we suggest a product-form formulation for the measurable general cost of a path.

$$G_r^w = C_r^w \left(1 + \eta\varphi_r^w\right) \ \forall r \in R^w, w \in W \tag{5}$$

where $\eta$ is a dimensionless positive number.

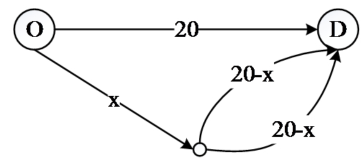


**Figure 1: The Testing Network (The Number in Each Edge Indicates Its Physic Length, and 0≤x≤20).**

Substituting formula (5) into formulas (3) and (4) gives the product-form C-Logit model. The complete model formulation of the product-form C-logit-SUE model is given below:

$$f_r^w = \frac{\exp\left(-\theta C_r^w \left(1 + \eta\varphi_r^w\right)\right)}{\sum_{i \in R^w} \exp\left(-\theta C_i^w \left(1 + \eta\varphi_i^w\right)\right)} \cdot q^w \ \forall r \in R^w, w \in W \tag{6}$$

where

$$\sum\nolimits_{r \in R^w} f_r^w = q^w \ \forall w \in W; \ f_r^w \geq 0 \ \forall k \in K^w, w \in W \tag{7}$$

In formula (7), the first expression shows the conservation between path flows and travel demands, and the second one is the non-negative constraint for path flow.

In contrast with the traditional model, the overlap loss of the product-form C-logit model can be understood as $\eta\varphi_r^w C_r^w$. Obviously, it is no longer a constant, which is expected to avoid degradation. This remark will be discussed through numerical experiments in next section.

## 3 NUMERICAL EXPERIMENTS

In this section, we discuss the effectiveness of the product-form C-logit model through numerical experiments. To this end, the assignment results between the traditional and the current C-logit model will be compared. For simplicity and to focus on the essence, here we do not perform the experiments on a complex example network but just on a specially designed simple one shown in Figure 1 which contains 1 OD pair (begins from node O and ends at node D), 3 nodes, 4 edges and 3 paths (where path 1 is referred to the independent one, while path 2 and path 3 are partially overlapped). For brevity, we omit the notational units in the following statement.

In this network, the free-flow travel time of each edge is proportional to physical length. Let the free-flow travel time of path 1 be 20. The capacity of each edge is set to be 25. In this example, the travel time of each edge is calculated by the BPR function $t_a = t_{a0} \times (1 + 0.15(\frac{v_a}{Y_a})^4)$ where $t_a$ is the actual travel time through edge $a$, $t_{a0}$ is the free-flow travel time, $v_a$ is the traffic flow of edge $a$, and $Y_a$ is the designed capacity of road section $a$.

For two C-logit-SUE models, we adopt the path correlation measure developed in Zhang et al. [12] for calculating the path overlap degree, and the concrete formulation is omitted here due to limited space. We solve the two models by the method of successive average [6], and readers are referred to, e.g., [6] and [12] for the algorithmic procedure.

Figures 2, 3 and 4 display the traffic assignment results of path 1 obtained from two C-logit models when x=5, 10, and 15, respectively. Note that in each subfigure contained in Figures 2, 3 and 4, the

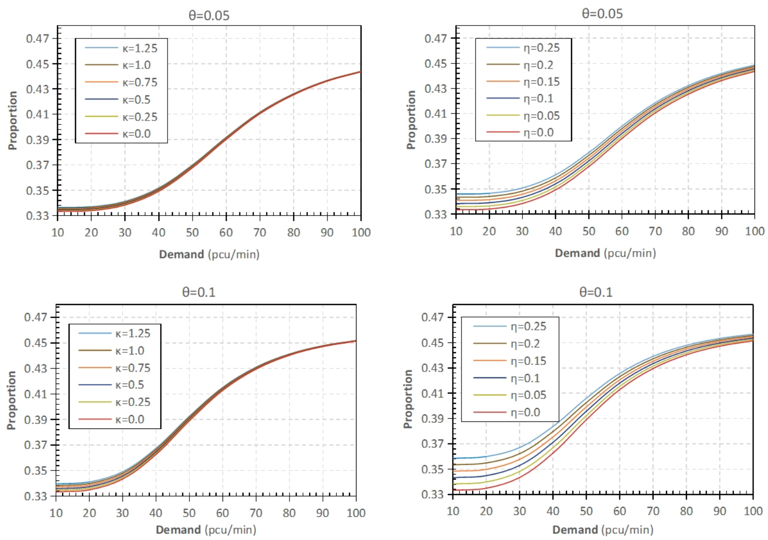


**Figure 2: The Traffic Proportion on Path 1 for Traditional C-Logit Model (Left Column) and Product-Form C-Logit Model (Right Column) When X=5.**

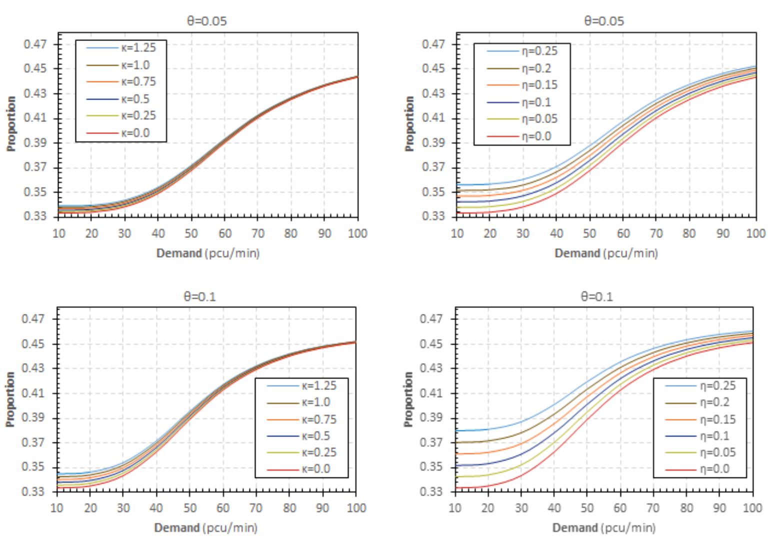


**Figure 3: The Traffic Proportion on Path 1 for Traditional C-logit Model (left column) and Product-form C-logit Model (right column) when x=10.**

curves with $\kappa = 0$ or $\eta = 0$ indicate the solutions of the classical logit model, which exactly corresponds to the degradation case.

It can be observed from the longitudinal comparison of the above figures that given $\kappa$ and $\eta$, the traffic assigned onto path 1 increases as $\theta$ grows. This is because path 1 does not overlap the other two paths, but path 2 and path 3 overlap and would suffer from overlap loss. Then, the growth of $\theta$ would enlarge the dominance of path 1 relative to path 2 and path 3, leading to an increased traffic proportion on path 1. Similarly, the growth of x intensifies the overlap between path 2 and path 3, which also raises the comparative dominance of path 1 and increases its traffic proportion hence.

It is found from horizontal comparison of the above figures that given a specific $\theta$, the traffic flow assigned onto path 1 rises as $\kappa$ and $\eta$ grow. This is because the growth of either $\kappa$ or $\eta$ implies increasing overlap loss for the corresponding C-logit model. In spite of this, the traffic growth (on path 1) resulted from increasing $\eta$ is much

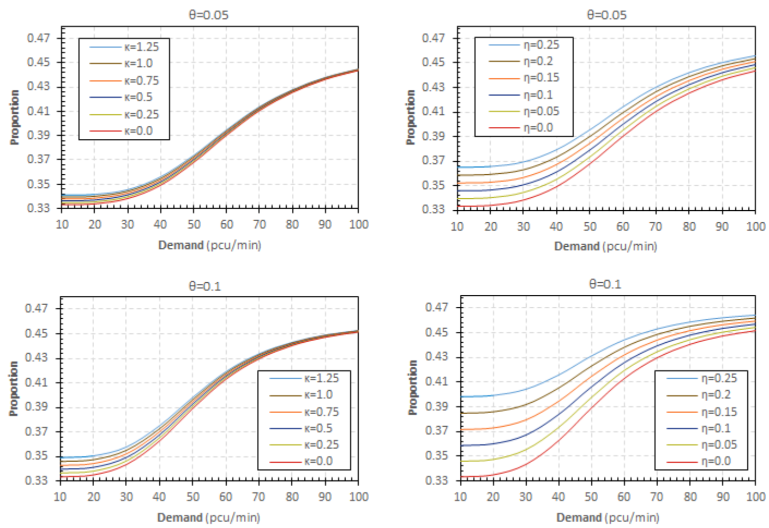


**Figure 4: The Traffic Proportion on Path 1 for Traditional C-logit Model (left column) and Product-form C-logit Model (right column) when x=15.**

more significant than that from increasing $\kappa$. These demonstrate that the product-form C-logit model shows far higher sensitivity to varying traffic condition than the traditional C-logit model.

Taken together, two general observations could be obtained. First, despite the fast growth of $\kappa$, the degradation of traditional C-logit model still emerges even though travel demands are not large, which implies that the traditional C-logit model can only overcome the shortcoming of the logit model caused by IIA to a quite limited extent. Second, with the growth of travel demand, the solutions of the product-form model consistently maintain observable gaps from those of the logit model, and those gaps enlarge as either $\theta$ or x increase. This demonstrates that the current product-form formulation for the measurable general path cost is effective, which endows the new C-logit model with clearly enhanced capacity to resist degradation.

## 4 CONCLUDING REMARKS

This paper suggests a product-form C-logit model for stochastic traffic assignment based on the traditional C-logit model. It is found from the numerical example that such a slightly revised C-logit model shows much higher sensitivity to varying traffic condition than the traditional model, and thus possesses much stronger resistance to degradation into the logit model.

The current study can be extended from different directions. To name a few examples, first, the product-form C-logit model needs to be validated and calculated via empirical data. Second, the equivalent nonlinear programming model is encouraged to be developed, and more efficient algorithms could be explored for solving the model in large or real-life road networks.

## ACKNOWLEDGMENTS

This study is jointly supported by the National Key Research and Development Program of China (2019YFF0301300), and the Fundamental Research Funds for the Central Universities (2019JBM038).

## REFERENCES

[1] J G Wardrop (1952). Some theoretical aspects of road traffic research. Proceedings of the Institution of Civil Engineers, Part II-Research and Theory, 1(36), 325-362.
[2] M J Beckmann, C B McGuire and C B Winston (1956). Studies in the Economics of Transportation. Yale University Press, New Haven, CT, USA.
[3] D E Boyce, H S Mahmassani and A Nagurney (2015). A retrospective on BMW's studies in the economics of transportation. Journal of Regional Science, 84(1), 85–103.
[4] C F Daganzo and Y Sheffi (1977). On stochastic models of traffic assignment, Transportation Science, 11(3), 253-274.
[5] M. Ben-akiva and S R Lerman (1985). Discrete Choice Analysis: Theory and Application to Travel Demand. MIT Press, Cambridge, MA, USA.
[6] Y Sheffi (1985). Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods. Prentice-Hall, Englewood Cliffs, New Jersey, USA.
[7] E Cascetta (1989). A stochastic process approach to the analysis of temporal dynamics in transportation networks. Transportation Research Part B, 23(1), 1-17.
[8] Z Zhou, A Chen and S Bekhor (2012). C-logit stochastic user equilibrium model: formulations and solution algorithm. Transportmetrica, 8(1), 17–41.
[9] M Ben-Akiva and M Bierlaire (1999). Discrete choice methods and their applications to short term travel decisions. In: R W Hall (eds.). Handbook of Transportation Science. Springer, Boston, MA, USA. pp. 5-33.
[10] P Vovsha (1997). Application of cross-nested logit model to mode choice in tel aviv, israel, metropolitan area. Transportation Research Record, 1607(1), 6-15.
[11] F S Koppelman and C H Wen (2000). The paired combinatorial logit model: properties, estimation and application. Transportation Research Part B, 34(2), 75-89.
[12] W Zhang, Z Li, H Li and H Zhong (2018). A probability-dominant C-logit stochastic traffic assignment model. Journal of Beijing Jiaotong University, 42(06), 61-66. (in Chinese).